$B(0, N)$-graded Lie superalgebras coordinatized by quantum tori

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Abstract

We use a fermionic extension of the bosonic module to obtain a class of $B(0, N)$-graded Lie superalgebras with nontrivial central extensions.

0 Introduction

$B(M-1, N)$-graded Lie superalgebras were first investigated and classified up to central extension by Benkart-Elduque (see also Garcia-Neher's work in [GN]). Those root graded Lie superalgebras are a super-analog of root graded Lie algebras. Fermionic and bosonic representations for the affine Kac-Moody Lie algebras were studied by Frenkel [F1,2] and Kac-Peterson [KP]. Feingold-Frenkel [FF] constructed representations for all classical affine Lie algebras by using Clifford or Weyl algebras with infinitely many generators. They also obtained realizations for certain affine Lie superalgebras including the affine $B(0, N)$. [G] gave bosonic and fermionic representations for the extended affine Lie algebra $\widetilde{gl_N(\mathbb{C}_q)}$, where $\mathbb{C}_q$ is the quantum torus in two variables. [CG] constructed modules for some BC_N-graded Lie algebras by considering a fermionic extension of the fermionic module.

In this paper, we will consider a fermionic extension of the bosonic module to obtain a class of $B(0, N)$-graded Lie superalgebras with nontrivial central extensions.

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The organization of the paper is as follows. In Section 1, we review some basics on the quantum torus and present examples of $B(0,N)$-graded Lie superalgebras coordinatized by quantum tori which are subalgebras of $\widehat{gl(1,2N)}(\mathbb{C}_q)$. In Section 2, we use bosons and fermions to construct representations for those examples of $B(0,N)$-graded Lie superalgebras.

Throughout this paper, we denote the field of complex numbers and the ring of integers by $\mathbb{C}$ and $\mathbb{Z}$ respectively.

1 B(0,N)-graded Lie superalgebras

We first recall some basics on quantum tori and then go on to present examples of B(0,N)-graded Lie superalgebras coordinatized by quantum tori. For more information on Lie superalgebras graded by root systems, see [BE1]-[BE2] and [GN].

Let q be a non-zero complex number. A quantum torus associated to q (see [M]) is the unital associative $\mathbb{C}$-algebra $\mathbb{C}_q[x^{\pm}, y^{\pm}]$(or, simply $\mathbb{C}_q$) with generators $x^{\pm}$, $y^{\pm}$ and relations

$$xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1 \quad and \quad yx = qxy. \tag{1.1}$$

Set $\Lambda(q) = \{n \in \mathbb{Z} | q^n = 1\}$. From [BGK] we see that $[\mathbb{C}_q, \mathbb{C}_q]$ has a basis consisting of monomials $x^m y^n$ for $m \notin \Lambda(q)$ or $n \notin \Lambda(q)$.

Let $^-$ be the anti-involution on $\mathbb{C}_q$ given by

$$\bar{x} = x, \quad \bar{y} = y^{-1}. \tag{1.2}$$

We have $\mathbb{C}_q = \mathbb{C}_q^+ \oplus \mathbb{C}_q^-$, where $\mathbb{C}_q^{\pm} = \{s \in \mathbb{C}_q | \bar{s} = \pm s\}$, then

$$\begin{aligned} \mathbb{C}_q^+ &= span\{x^m y^n + \overline{x^m y^n} | m \in \mathbb{Z}, n \geq 0\}, \\ \mathbb{C}_q^- &= span\{x^m y^n - \overline{x^m y^n} | m \in \mathbb{Z}, n > 0\}. \end{aligned} \tag{1.3}$$

Let M, N be two positive integers. We have a Lie superalgebra $gl(M,N)(\mathbb{C}_q)$ of $(M+N)$ by $(M+N)$ matrices with entries from $\mathbb{C}_q$.

We form a central extension of Lie superalgebra $gl(M,N)(\mathbb{C}_q)$ as was done in [G] and [CG].

$$\widehat{gl(M,N)}(\mathbb{C}_q) = gl(M,N)(\mathbb{C}_q) \oplus \Big(\sum_{n \in \Lambda(q)} \oplus \mathbb{C} c(n) \Big) \oplus \mathbb{C} c_y \tag{1.4}$$

with bracket

$$[A(x^my^n), B(x^py^s)]_s = A(x^my^n)B(x^py^s) - (-1)^{degAdegB}B(x^py^s)A(x^my^n) \tag{1.5}$$
$$+mq^{np}str(AB)\delta_{m+p,0}\delta_{\overline{n+s},\bar{0}}c(n+s) + nq^{np}str(AB)\delta_{m+p,0}\delta_{n+s,0}c_y$$

for $m, p, n, s \in \mathbb{Z}$, $A, B \in gl(M,N)_\alpha, \alpha = \bar{0}$ or $\bar{1}$, where str is the super-trace of the Lie superalgebra $gl(M,N)$, $c(u)$ with $u \in \Lambda(q)$ and c_y are central elements of $\widehat{gl(M,N)}(\mathbb{C}_q)$, $\bar{t}$ means $\bar{t} \in \mathbb{Z}/\Lambda(q)$, for $t \in \mathbb{Z}$.

Now we present the examples of Lie superalgebra graded by the root system of type $B(0,N)$. We first set

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -I_{2N} \end{pmatrix}, G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_N \\ 0 & -I_N & 0 \end{pmatrix} \in M_{2N+1}(\mathbb{C}_q).$$

Then, G and J are invertible $(2N+1)\times(2N+1)$-matrices. Using the matrix G and J, we define a superspace $\mathcal{S}$ with:

$$\mathcal{S}_{\bar{0}} = \{X \in gl(1,2N)(\mathbb{C}_q)_{\bar{0}} | \bar{X}^tG + GX = 0\}$$

$$\mathcal{S}_{\bar{1}} = \{X \in gl(1,2N)(\mathbb{C}_q)_{\bar{1}} | \bar{X}^tG - JGX = 0\}$$

We can easily see that $\mathcal{S}$ is a subalgebra of $gl(1,2N)(\mathbb{C}_q)$ over $\mathbb{C}$. The general form of a matrix in $\mathcal{S}$ is

$$\begin{pmatrix} a & b & c \\ \bar{c}^t & A & S \\ -\bar{b}^t & T & -\bar{A}^t \end{pmatrix} \quad \text{with } \bar{a} = -a \quad \bar{S}^t = S \quad \text{and} \quad \bar{T}^t = T, \tag{1.6}$$

where A, S, T are $N\times N$ sub-matrices. Then the Lie superalgebra $\mathcal{G} = [\mathcal{S},\mathcal{S}]_s$, is a B(0,N)-graded Lie superalgebra.

As in [AABGP], we easily know that:

$$\mathcal{G} = \{Y \in gl(1,2N)(\mathbb{C}_q) | str(Y) \equiv 0 \text{ mod } [\mathbb{C}_q, \mathbb{C}_q]\}.$$

Putting

$$\mathcal{H} = \Big\{\sum_{i=1}^{N} a_i(e_{ii} - e_{N+i,N+i}) | a_i \in \mathbb{C}\Big\}, \tag{1.7}$$

then $\mathcal{H}$ is a N-dimensional abelian subalgebra of $\mathcal{G}$. Defining $\delta_i \in \mathcal{H}^*, i = 1, \cdots, N$, by

$$\delta_i\Big(\sum_{j=1}^{N} a_j(e_{jj} - e_{N+j,N+j})\Big) = a_i \tag{1.8}$$

for $i = 1, \cdots, N$. Setting $\mathcal{G}_\alpha = \{x \in \mathcal{G} | [h, x]_s = \alpha(h)x, \text{ for all } h \in \mathcal{H}\}$ as usual, we have

$$\mathcal{G} = \mathcal{G}_0 \oplus \sum_{i\neq j} \mathcal{G}_{\delta_i-\delta_j} \oplus \sum_{i<j}(\mathcal{G}_{\delta_i+\delta_j} \oplus \mathcal{G}_{-\delta_i-\delta_j}) \oplus \sum_i (\mathcal{G}_{\delta_i} \oplus \mathcal{G}_{-\delta_i} \oplus \mathcal{G}_{2\delta_i} \oplus \mathcal{G}_{-2\delta_i}), \tag{1.9}$$

where

$$\begin{aligned}
\mathcal{G}_{\delta_i-\delta_j} &= span_{\mathbb{C}}\{\tilde{f}_{ij}(m,n) = x^m y^n e_{ij} - \overline{x^m y^n} e_{N+j,N+i} | m, n \in \mathbb{Z}\},\\
\mathcal{G}_{\delta_i+\delta_j} &= span_{\mathbb{C}}\{\tilde{g}_{ij}(m,n) = x^m y^n e_{i,N+j} + \overline{x^m y^n} e_{j,N+i} | m, n \in \mathbb{Z}\},\\
\mathcal{G}_{-\delta_i-\delta_j} &= span_{\mathbb{C}}\{\tilde{h}_{ij}(m,n) = -x^m y^n e_{N+i,j} - \overline{x^m y^n} e_{N+j,i} | m, n \in \mathbb{Z}\},\\
\mathcal{G}_{2\delta_i} &= span_{\mathbb{C}}\{\tilde{g}_{ii}(m,n) = (x^m y^n + \overline{x^m y^n}) e_{i,N+i} | m, n \in \mathbb{Z}\},\\
\mathcal{G}_{-2\delta_i} &= span_{\mathbb{C}}\{\tilde{h}_{ii}(m,n) = -(x^m y^n + \overline{x^m y^n}) e_{N+i,i} | m, n \in \mathbb{Z}\},\\
\mathcal{G}_{\delta_i} &= span_{\mathbb{C}}\{\tilde{e}_i(m,n) = -x^m y^n e_{i,0} - \overline{x^m y^n} e_{0,N+i} | m, n \in \mathbb{Z}\},\\
\mathcal{G}_{-\delta_i} &= span_{\mathbb{C}}\{\tilde{e}_i^*(m,n) = x^m y^n e_{N+i,0} - \overline{x^m y^n} e_{0,i} | m, n \in \mathbb{Z}\}
\end{aligned} \tag{1.10}$$

and

$$\mathcal{G}_0 = span_{\mathbb{C}}\{\tilde{f}_{ii}(m,n) + \tilde{e}_0(m,n), \tilde{e}_0(p,s) | 1 \leq i \leq N, m, n \in \mathbb{Z}, p \notin \Lambda(q) \text{ or } s \notin \Lambda(q)\}$$

where $\tilde{e}_0(m,n) = -(x^m y^n - \overline{x^m y^n}) e_{0,0}$.

We then have a central extension of $\mathcal{G}$

$$\widehat{\mathcal{G}} = \mathcal{G} \oplus \Big(\sum_{n\in\Lambda(q)} \oplus \mathbb{C}c(n)\Big) \oplus \mathbb{C}c_y \tag{1.11}$$

with bracket as (1.5).

We have

Proposition 1.1

$$[\tilde{g}_{ij}(m,n), \tilde{g}_{kl}(p,s)]_s = 0 \tag{1.12}$$

(1.13)
$$[\tilde{g}_{ij}(m,n),\tilde{f}_{kl}(p,s)]_s=-\delta_{il}q^{ms}\tilde{g}_{kj}(m+p,n+s)-\delta_{jl}q^{(s-n)m}\tilde{g}_{ki}(m+p,s-n)$$

$$\begin{aligned}
&[\tilde{g}_{ij}(m,n),\tilde{h}_{kl}(p,s)]_s\\
=\;&-\delta_{ik}q^{-n(m+p)}\tilde{f}_{jl}(m+p,s-n)-\delta_{jk}q^{np}\tilde{f}_{il}(m+p,n+s)\\
&-\delta_{il}q^{-(mn+np+ps)}\tilde{f}_{jk}(m+p,-(n+s))-\delta_{jl}q^{(n-s)p}\tilde{f}_{ik}(m+p,n-s)\\
&+mq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}(c(n+s)+c(-n-s))\\
&+m\delta_{ik}\delta_{jl}\delta_{m+p,0}\delta_{\overline{n-s},\overline{0}}(c(n-s)+c(s-n))
\end{aligned} \tag{1.14}$$

$$[\tilde{g}_{ij}(m,n),\tilde{e}_k(p,s)]_s=[\tilde{g}_{ij}(m,n),\tilde{e}_0(p,s)]_s=0 \tag{1.15}$$

(1.16)
$$[\tilde{g}_{ij}(m,n),\tilde{e}^*_k(p,s)]_s=-\delta_{ik}q^{-n(m+p)}\tilde{e}_j(m+p,s-n)-\delta_{jk}q^{np}\tilde{e}_i(m+p,n+s)$$

$$[\tilde{g}_{ij}(m,n),\tilde{e}_0(p,s)]_s=0 \tag{1.17}$$

$$\begin{aligned}
[\tilde{f}_{ij}(m,n),\tilde{f}_{kl}(p,s)]_s\;=\;&\delta_{jk}q^{np}\tilde{f}_{il}(m+p,n+s)-\delta_{il}q^{sm}\tilde{f}_{kj}(m+p,n+s)\\
&-2mq^{np}\delta_{jk}\delta_{il}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}c(n+s)
\end{aligned} \tag{1.18}$$

(1.19)
$$[\tilde{f}_{ij}(m,n),\tilde{h}_{kl}(p,s)]_s=-\delta_{ik}q^{-n(m+p)}\tilde{h}_{jl}(m+p,s-n)-\delta_{il}q^{ms}\tilde{h}_{kj}(m+p,n+s)$$

$$[\tilde{f}_{ij}(m,n),\tilde{e}_k(p,s)]_s=\delta_{jk}q^{np}\tilde{e}_i(m+p,n+s) \tag{1.20}$$

$$[\tilde{f}_{ij}(m,n),\tilde{e}^*_k(p,s)]_s=-\delta_{ik}q^{-n(m+p)}\tilde{e}^*_j(m+p,s-n) \tag{1.21}$$

$$[\tilde{f}_{ij}(m,n),\tilde{e}_0(p,s)]_s=0 \tag{1.22}$$

$$(1.23)\qquad [\tilde{h}_{ij}(m,n),\tilde{h}_{kl}(p,s)]_s=0$$

(1.24)
$$[\tilde{h}_{ij}(m,n),\tilde{e}_k(p,s)]_s=\delta_{jk}q^{np}\tilde{e}_i^*(m+p,n+s)+\delta_{ik}q^{-n(m+p)}\tilde{e}_j^*(m+p,s-n)$$

$$(1.25)\qquad [\tilde{h}_{ij}(m,n),\tilde{e}_k^*(p,s)]_s=0$$

$$(1.26)\qquad [\tilde{h}_{ij}(m,n),\tilde{e}_0(p,s)]_s=0$$

$$(1.27)\qquad [\tilde{e}_i(m,n),\tilde{e}_k(p,s)]_s=q^{m(s-n)}\tilde{g}_{ki}(m+p,s-n)$$

$$(1.28)\qquad \begin{aligned} &[\tilde{e}_i(m,n),\tilde{e}_k^*(p,s)]_s\\ =\ &\delta_{ik}q^{-n(m+p)}\tilde{e}_0(m+p,s-n)+q^{p(n-s)}\tilde{f}_{ik}(m+p,n-s)\\ &-m\delta_{ik}\delta_{m+p,0}\delta_{\overline{n-s},\overline{0}}(c(n-s)+c(s-n))\end{aligned}$$

$$(1.29)\quad [\tilde{e}_i(m,n),\tilde{e}_0(p,s)]_s=-q^{np}\tilde{e}_i(m+p,n+s)+q^{p(n-s)}\tilde{e}_i(m+p,n-s)$$

$$(1.30)\qquad [\tilde{e}_i^*(m,n),\tilde{e}_k^*(p,s)]_s=q^{m(s-n)}\tilde{h}_{ki}(m+p,s-n)$$

$$(1.31)\quad [\tilde{e}_i^*(m,n),\tilde{e}_0(p,s)]_s=-q^{np}\tilde{e}_i^*(m+p,n+s)+q^{p(n-s)}\tilde{e}_i^*(m+p,n-s)$$

$$(1.32)\quad \begin{aligned} &[\tilde{e}_0(m,n),\tilde{e}_0(p,s)]_s\\ =\ &-(q^{np}-q^{sm})\tilde{e}_0(m+p,n+s)-(q^{m(s-n)}-q^{-n(m+p)})\tilde{e}_0(m+p,s-n)\\ &+mq^{np}\delta_{m+p,0}\delta_{\overline{n+s},\overline{0}}(c(n+s)+c(-n-s))\\ &-m\delta_{m+p,0}\delta_{\overline{n-s},\overline{0}}(c(n-s)+c(s-n))\end{aligned}$$

for all $m,p,n,s\in\mathbb{Z}$ *and* $1\le i,j,k,l\le N$.

Remark 1.1 For our situation in the decomposition of $B(0,N)$-graded Lie superalgebras in §5[Be2], $A=\mathbb{C}_q^+$ and $B=\mathbb{C}_q^-$.

2 The module construction

In this section, we follow the method in [G] and [CG] to construct representations for the Lie superalgebras which are given in Section 1. The idea goes back to [FF].

Let $\mathcal{R}$ be an associative algebra. Let $\rho = \pm 1$. We define a ρ-bracket on $\mathcal{R}$ as follow:

$$\{a, b\}_\rho = ab + \rho ba, \quad a, b \in \Re. \tag{2.1}$$

It is easy to see that

$$\{a, b\}_\rho = \rho\{b, a\}_\rho \tag{2.2}$$

for $a, b, c \in \mathcal{R}$.

Define $\mathfrak{a}$ to be the unital associative algebra with $2N$ generators $a_i, a_i^*, 1 \leq i \leq N$, subject to relations

$$\{a_i, a_j\}_- = \{a_i^*, a_j^*\}_- = 0, \quad \text{and} \quad \{a_i, a_j^*\}_- = -\delta_{ij}. \tag{2.3}$$

Let the associative algebra $\alpha(N, -)$ be generated by

$$\{u(m) | u \in \bigoplus_{i=1}^{N} (\mathbb{C}a_i \oplus \mathbb{C}a_i^*), m \in \mathbb{Z}\} \tag{2.4}$$

with the relations

$$\{u(m), v(n)\}_- = \{u, v\}_- \delta_{m+n,0}. \tag{2.5}$$

The normal ordering is defined as in [FF](see also [F2] or [CG]).

$$\begin{aligned} : u(m)v(n) : \; &= \begin{cases} u(m)v(n) & \text{if } n > m, \\ \frac{1}{2}\big(u(m)v(n) + v(n)u(m)\big) & \text{if } n = m, \\ v(n)u(m) & \text{if } n < m, \end{cases} \\ &= \; : v(n)u(m) : \end{aligned} \tag{2.6}$$

for $n, m \in \mathbb{Z}, u, v \in \mathfrak{a}$.

Next we consider an extension of the algebra $\alpha(N, -)$. The generators

$$\{e(m) | m \in \mathbb{Z}\} \tag{2.7}$$

span an infinite-dimensional Clifford algebra with relations

$$\{e(m), e(n)\}_+ = e(m)e(n) + e(n)e(m) = -\delta_{n+m,0}. \tag{2.8}$$

Let $\alpha_\tau(N)$ denote the algebra obtained by adjoining to $\alpha(N,-)$ the generators (2.7) with relations (2.8) and

$$\{a_i(m), e(n)\}_\tau = 0 = \{a_i^*(m), e(n)\}_\tau, \text{ for } \tau = \pm 1. \tag{2.9}$$

The normal ordering is given as follows

$$\begin{gathered} : e(m)e(n) := \begin{cases} e(m)e(n) & \text{if } n > m, \\ \frac{1}{2}\big(e(m)e(n) - e(n)e(m)\big) & \text{if } n = m, \\ -e(n)e(m) & \text{if } n < m, \end{cases} , \\ : a_i(m)e(n) := a_i(m)e(n) = -\tau e(n)a_i(m), \\ : a_i^*(m)e(n) := a_i^*(m)e(n) = -\tau e(n)a_i^*(m), \end{gathered} \tag{2.10}$$

for $n, m \in \mathbb{Z}, 1 \le i, j \le N$. Set

$$\theta(n) = \begin{cases} 1, & \text{for } n > 0, \\ \frac{1}{2}, & \text{for } n = 0, \\ 0, & \text{for } n < 0, \end{cases} \quad \text{then } 1 - \theta(n) = \theta(-n). \tag{2.11}$$

We have

$$\begin{gathered} : a_i(m)a_j(n) := a_i(m)a_j(n) = a_j(n)a_i(m), \\ : a_i^*(m)a_j^*(n) := a_i^*(m)a_j^*(n) = a_j^*(n)a_i^*(m), \end{gathered} \tag{2.12}$$

and

$$\begin{gathered} a_i(m)a_j^*(n) =: a_i(m)a_j^*(n) : -\delta_{ij}\delta_{m+n,0}\theta(m-n), \\ a_j^*(n)a_i(m) =: a_i(m)a_j^*(n) : +\delta_{ij}\delta_{m+n,0}\theta(n-m), \\ e(m)e(n) =: e(m)e(n) : -\delta_{m+n,0}\theta(m-n). \end{gathered} \tag{2.13}$$

Let $\alpha(N,-)^+$ be the subalgebra generated by $a_i(n), a_j^*(m), a_k^*(0)$, for $n, m > 0$, and $1 \le i, j, k \le N$. Let $\alpha(N,-)^-$ be the subalgebra generated by $a_i(n), a_j^*(m), a_k(0)$, for $n, m < 0$, and $1 \le i, j, k \le N$. Those generators in $\alpha(N,-)^+$ are called annihilation operators while those in $\alpha(N,-)^-$ are called creation operators. Let $V(N,-)$ be a simple $\alpha(N,-)$-module containing an

element v_0, called a "vacuum vector", and satisfying

$$\alpha(N,-)^+ v_0 = 0. \tag{2.14}$$

So all annihilation operators kill v_0 and

$$V(N,-) = \alpha(N,-)^- v_0. \tag{2.15}$$

Let V_0 be a simple Clifford module for the Clifford algebra generated by (2.7) with relations (2.8) and containing "vacuum vector" v_0', which is killed by annihilation operators.(Here we call $e(m)$ annihilation operator if $m > 0$, or a creation operator if $m < 0$. e(0) acts as scalar.) Because of (2.9), we see that the $\alpha_\tau(N)$-module

$$V_\tau(N) = V(N,-) \otimes V_0 = \alpha_\tau(N) v_0' \tag{2.16}$$

is simple.

Now we define our operators on $V_\tau(N)$. For any $m, n \in \mathbb{Z}, 1 \leq i, j \leq N$, set

$$f_{ij}(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : a_i(m-s)a_j^*(s) : \tag{2.17}$$

$$g_{ij}(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : a_i(m-s)a_j(s) : \tag{2.18}$$

$$h_{ij}(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : a_i^*(m-s)a_j^*(s) : \tag{2.19}$$

$$e_i(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : a_i(m-s)e(s) : \tag{2.20}$$

$$e_i^*(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : a_i^*(m-s)e(s) : \tag{2.21}$$

$$e_0(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} : e(m-s)e(s) : . \tag{2.22}$$

We will list only those commutation relations involving $e(s)$ (See [CG]).

Lemma 2.1 *We have*

$$[a_i(m)a_j(n), a_k(p)e(s)] = [a_i(m)a_j(n), e(p)e(s)] = 0 \tag{2.23}$$

$$[a_i(m)a_j(n), a_k^*(p)e(s)] = -\delta_{ik}\delta_{m+p,0}a_j(n)e(s) - \delta_{jk}\delta_{n+p,0}a_i(m)e(s) \tag{2.24}$$

$$[a_i(m)a_j^*(n), a_k(p)e(s)] = \delta_{jk}\delta_{n+p,0}a_i(m)e(s), \tag{2.25}$$

$$[a_i(m)a_j^*(n), a_k^*(p)e(s)] = -\delta_{ik}\delta_{m+p,0}a_j^*(n)e(s), \tag{2.26}$$

$$[a_i(m)a_j^*(n), e(p)e(s)] = [a_i^*(m)a_j^*(n), a_k^*(p)a_l^*(s)] = 0 \tag{2.27}$$

$$[a_i^*(m)a_j^*(n), a_k(p)e(s)] = \delta_{jk}\delta_{n+p,0}a_i^*(m)e(s) + \delta_{ik}\delta_{m+p,0}a_j^*(m)e(s), \tag{2.28}$$

$$[a_i^*(m)a_j^*(n), a_k^*(p)e(s)] = [a_i^*(m)a_j^*(n), e(p)e(s)] = 0 \tag{2.29}$$

$$\{a_i(m)e(n), a_k(p)e(s)\}_+ = \tau\delta_{n+s,0}a_i(m)a_k(p) \tag{2.30}$$

$$\{a_i(m)e(n), a_k^*(p)e(s)\}_+ = \tau\delta_{n+s,0}a_k^*(p)a_i(m) + \tau\delta_{ik}\delta_{m+p,0}e(n)e(s) \tag{2.31}$$

$$[a_i(m)e(n), e(p)e(s)] = \delta_{n+s,0}a_i(m)e(p) - \delta_{n+p,0}a_i(m)e(s) \tag{2.32}$$

$$\{a_i^*(m)e(n), a_k^*(p)e(s)\}_+ = \tau\delta_{n+s,0}a_i^*(m)a_k^*(p) \tag{2.33}$$

$$[a_i^*(m)e(n), e(p)e(s)] = \delta_{n+s,0}a_i^*(m)e(p) - \delta_{n+p,0}a_i^*(m)e(s) \tag{2.34}$$

$$\begin{aligned} (2.35)\quad & [e(m)e(n), e(p)e(s)] \\ = \quad & -\delta_{n+p,0}e(m)e(s) + \delta_{m+p,0}e(n)e(s) - \delta_{n+s,0}e(p)e(m) + \delta_{m+s,0}e(p)e(n) \end{aligned}$$

for m,n,p,$s \in \mathbb{Z}$ *and* $1 \leq i, j, k, l \leq N$.

Proof We only check some of them.

$$
\begin{aligned}
& \{a_i(m)e(n), a_k(p)e(s)\}_+ = a_i(m)e(n)a_k(p)e(s) + a_k(p)e(s)a_i(m)e(n) \\
= \ & -\tau\big(a_i(m)a_k(p)e(n)e(s) + a_k(p)a_i(m)e(s)e(n)\big) \\
= \ & -\tau\big(a_i(m)a_k(p)(e(n)e(s) + e(s)e(n))\big) \\
= \ & \tau\delta_{n+s,0}a_i(m)a_k(p);
\end{aligned}
$$

$$
\begin{aligned}
& \{a_i(m)e(n), a_k^*(p)e(s)\}_+ = a_i(m)e(n)a_k^*(p)e(s) + a_k^*(p)e(s)a_i(m)e(n) \\
= \ & -\tau\big(a_i(m)a_k^*(p)e(n)e(s) + a_k^*(p)a_i(m)e(s)e(n)\big) \\
= \ & -\tau\big(a_k^*(p)a_i(m) - \delta_{ik}\delta_{m+p,0}\big)e(n)e(s) - \tau a_k^*(p)a_i(m)e(s)e(n) \\
= \ & \tau\delta_{n+s,0}a_k^*(p)a_i(m) + \tau\delta_{ik}\delta_{m+p,0}e(n)e(s);
\end{aligned}
$$

$$
\begin{aligned}
& [a_i(m)e(n), e(p)e(s)] = a_i(m)e(n)e(p)e(s) - e(p)e(s)a_i(m)e(n) \\
= \ & a_i(m)\big(e(n)e(p)e(s) - (-\tau)^2e(p)e(s)e(n)\big) \\
= \ & a_i(m)\big(-\delta_{n+p,0}e(s) - e(p)e(n)e(s) - e(p)e(s)e(n)\big) \\
= \ & a_i(m)\big(\delta_{n+s,0}e(p) - \delta_{n+p,0}e(s)\big) \\
= \ & \delta_{n+s,0}a_i(m)e(p) - \delta_{n+p,0}a_i(m)e(s);
\end{aligned}
$$

$$
\begin{aligned}
& [e(m)e(n), e(p)e(s)] = e(m)e(n)e(p)e(s) - e(p)e(s)e(m)e(n) \\
= \ & e(m)(-\delta_{n+p,0} - e(p)e(n))e(s) - e(p)e(s)e(m)e(n) \\
= \ & -\delta_{n+p,0}e(m)e(s) - e(m)e(p)e(n)e(s) - e(p)e(s)e(m)e(n) \\
= \ & -\delta_{n+p,0}e(m)e(s) - (-\delta_{m+p,0} - e(p)e(m))e(n)e(s) - e(p)e(s)e(m)e(n) \\
= \ & -\delta_{n+p,0}e(m)e(s) + \delta_{m+p,0}e(n)e(s) + e(p)\big(e(m)e(n)e(s) - e(s)e(m)e(n)\big) \\
= \ & -\delta_{n+p,0}e(m)e(s) + \delta_{m+p,0}e(n)e(s) + e(p)\big(\delta_{m+s,0}e(n) - \delta_{n+s,0}e(m)\big) \\
= \ & -\delta_{n+p,0}e(m)e(s) + \delta_{m+p,0}e(n)e(s) - \delta_{n+s,0}e(p)e(m) + \delta_{m+s,0}e(p)e(n).
\end{aligned}
$$

So (2.30),(2.31),(2.32) and (2.35) hold true. ∎

In what follows we shall mean $\frac{q^{mn}-1}{q^n-1} = m$ if $n \in \Lambda(q)$. This will make our formula more concise.

Next we list all brackets that are needed. For all $m, p, n, s \in \mathbb{Z}$ and $1 \le i, j, k, l \le N$,

Proposition 2.1

$$[g_{ij}(m,n), g_{kl}(p,s)] = 0$$

Proposition 2.2

$$[g_{ij}(m,n), f_{kl}(p,s)] = -\delta_{il} q^{ms} g_{kj}(m+p, n+s) - \delta_{jl} q^{(s-n)m} g_{ki}(m+p, s-n)$$

Proposition 2.3

$$\begin{aligned} & [g_{ij}(m,n), h_{kl}(p,s)] \\ = \; & -\delta_{ik} q^{-n(m+p)} f_{jl}(m+p, s-n) - \delta_{jk} q^{np} f_{il}(m+p, n+s) \\ & -\delta_{il} q^{-(mn+np+ps)} f_{jk}(m+p, -(n+s)) - \delta_{jl} q^{(n-s)p} f_{ik}(m+p, n-s) \\ & +\delta_{ik}\delta_{jl}\delta_{m+p,0} \frac{1}{2}(q^{s-n}+1) \frac{q^{m(s-n)}-1}{q^{s-n}-1} + \delta_{jk}\delta_{il}\delta_{m+p,0} q^{np} \frac{1}{2}(q^{s+n}+1) \frac{q^{m(s+n)}-1}{q^{s+n}-1} \end{aligned}$$

Proposition 2.4

$$[g_{ij}(m,n), e_k(p,s)] = [g_{ij}(m,n), e_0(p,s)] = 0$$

$$[g_{ij}(m,n), e_k^*(p,s)] = -\delta_{ik} q^{-n(m+p)} e_j(m+p, s-n) - \delta_{jk} q^{np} e_i(m+p, n+s)$$

Proposition 2.5

$$\begin{aligned} [f_{ij}(m,n), f_{kl}(p,s)] \;=\; & \delta_{jk} q^{np} f_{il}(m+p, n+s) - \delta_{il} q^{sm} f_{kj}(m+p, n+s) \\ & -\delta_{jk}\delta_{il} q^{np} \delta_{m+p,0} \frac{1}{2}(q^{s+n}+1) \frac{q^{m(s+n)}-1}{q^{s+n}-1} \end{aligned}$$

Proposition 2.6

$$[f_{ij}(m,n), h_{kl}(p,s)] = -\delta_{ik} q^{-n(m+p)} h_{jl}(m+p, s-n) - \delta_{il} q^{ms} h_{kj}(m+p, n+s)$$

Proposition 2.7

$$[f_{ij}(m,n), e_k(p,s)] = \delta_{jk} q^{np} e_i(m+p, n+s)$$

$$[f_{ij}(m,n), e_k^*(p,s)] = -\delta_{ik} q^{-n(m+p)} e_j^*(m+p, s-n)$$

$$[f_{ij}(m,n), e_0(p,s)] = 0$$

Proposition 2.8

$$[h_{ij}(m,n), h_{kl}(p,s)] = 0$$

Proposition 2.9

$$[h_{ij}(m,n), e_k(p,s)] = \delta_{jk} q^{np} e_i^*(m+p, n+s) + \delta_{ik} q^{-n(m+p)} e_j^*(m+p, s-n)$$

$$[h_{ij}(m,n), e_k^*(p,s)] = [h_{ij}(m,n), e_0(p,s)] = 0$$

Proposition 2.10

$$\{e_i(m,n), e_k(p,s)\}_+ = \tau q^{m(s-n)} g_{ki}(m+p, s-n)$$

$$\begin{gathered}\{e_i(m,n), e_k^*(p,s)\}_+ = \tau \delta_{ik} q^{-n(m+p)} e_0(m+p, s-n) + \tau q^{p(n-s)} f_{ik}(m+p, n-s) \\ -\tau \delta_{ik} \delta_{m+p,0} \frac{1}{2}(q^{s-n}+1) \frac{q^{m(s-n)}-1}{q^{s-n}-1}\end{gathered}$$

$$[e_i(m,n), e_0(p,s)] = -q^{np} e_i(m+p, n+s) + q^{p(n-s)} e_i(m+p, n-s)$$

Proposition 2.11

$$\{e_i^*(m,n), e_k^*(p,s)\}_+ = \tau q^{m(s-n)} h_{ki}(m+p, s-n)$$

$$[e_i^*(m,n), e_0(p,s)] = -q^{np} e_i^*(m+p, n+s) + q^{p(n-s)} e_i^*(m+p, n-s)$$

Proposition 2.12

$$\begin{aligned} & [e_0(m,n), e_0(p,s)] \\ = \; & -(q^{np} - q^{sm}) e_0(m+p, n+s) + \delta_{m+p,0} q^{np} \frac{1}{2}(q^{n+s}+1) \frac{q^{m(n+s)}-1}{q^{n+s}-1} \\ & -(q^{m(s-n)} - q^{-n(m+p)}) e_0(m+p, s-n) - \delta_{m+p,0} \frac{1}{2}(q^{s-n}+1) \frac{q^{m(s-n)}-1}{q^{s-n}-1} \end{aligned}$$

We only give proofs for 2.10 and 2.12. The proof for the others is either similar or easy.

Proofs of 2.10 and 2.12:

Note that from [CG], we have

$$\sum_{t \in \mathbb{Z}} q^{-xt} \Big(\theta(-2t) - \theta(-2m-2t) \Big) = \frac{1}{2}(q^x+1) \frac{q^{mx}-1}{q^x-1}. \tag{2.36}$$

By (2.30),(2.31),(2.32) and (2.35), we have

$$\{e_i(m,n), a_k(p)e(s)\}_+ = \tau q^{ns} a_k(p) a_i(m+s), \tag{2.37}$$

$$\{e_i(m,n), a_k^*(p)e(s)\}_+ = \tau q^{ns} a_k^*(p) a_i(m+s) + \tau \delta_{ik} q^{-n(m+p)} e(m+p)e(s), \tag{2.38}$$

$$[e_i(m,n), e(p)e(s)] = q^{ns} a_i(m+s)e(p) - q^{np} a_i(m+p)e(s), \tag{2.39}$$

and

$$[e_0(m,n), e(p)e(s)] = -(q^{np} - q^{-n(m+p)})e(m+p)e(s) - (q^{ns} - q^{-n(m+s)})e(p)e(m+s), \tag{2.40}$$

so we get

$$\begin{aligned} & \{e_i(m,n), q^{-st} : a_k(p-t)e(t) :\}_+ = \tau q^{nt-st} a_k(p-t) a_i(m+t) \\ = \; & \tau q^{m(s-n)} q^{-(s-n)(m+t)} a_k(p-t) a_i(m+t) \\ = \; & \tau q^{m(s-n)} q^{-(s-n)(m+t)} : a_k(p-t) a_i(m+t) :, \end{aligned}$$

$$\begin{aligned} & \{e_i(m,n), q^{-st} : a_k^*(p-t)e(t) :\}_+ \\ = \; & q^{-st}\Big(\tau \delta_{ik} q^{-n(m+p-t)} e(m+p-t)e(t) + \tau q^{nt} a_k^*(p-t) a_i(m+t)\Big) \\ = \; & \tau \delta_{ik} q^{-n(m+p)} q^{-(s-n)t} e(m+p-t)e(t) + \tau q^{-(s-n)t} a_k^*(p-t) a_i(m+t) \\ = \; & \tau \delta_{ik} q^{-n(m+p)} q^{-(s-n)t}\Big(: e(m+p-t)e(t) : - \delta_{m+p,0}\theta(m+p-2t)\Big) \\ & \tau q^{-(s-n)t}\big(: a_i(m+t) a_k^*(p-t) : + \delta_{ik}\delta_{m+p,0}\theta(p-m-2t)\big) \\ = \; & \tau \delta_{ik} q^{-n(m+p)} q^{-(s-n)t} : e(m+p-t)e(t) : + \tau q^{p(n-s)} q^{-(n-s)(p-t)} : a_i(m+t) a_k^*(p-t) : \\ & - \tau \delta_{ik}\delta_{m+p,0} q^{-(s-n)t}\big(\theta(-2t) - \theta(-2m-2t)\big); \end{aligned}$$

$$\begin{aligned} & [e_i(m,n), q^{-st} : e(p-t)e(t) :] \\ = \; & q^{-st}\big(q^{n(p-t)} a_i(m+p-t)e(t) + q^{nt} a_i(m+t)e(p-t)\big) \\ = \; & -q^{np} q^{-(n+s)t} : a_i(m+p-t)e(t) : + q^{p(n-s)} q^{-(n-s)(p-t)} : a_i(m+t)e(p-t) : \end{aligned}$$

and

$$
\begin{aligned}
& [e_0(m,n), q^{-st} : e(p-t)e(t) :] \\
= & -q^{-st}\Big((q^{n(p-t)} - q^{-n(m+p-t)})e(m+p-t)e(t) - (q^{nt} - q^{-n(m+t)})e(p-t)e(m+t)\Big) \\
= & -q^{-st}(q^{n(p-t)} - q^{-n(m+p-t)})\big(: e(m+p-t)e(t) : -\delta_{m+p,0}\theta(m+p-2t)\big) \\
& -q^{-st}(q^{nt} - q^{-n(m+t)})\big(: e(p-t)e(m+t) : -\delta_{m+p,0}\theta(p-m-2t)\big) \\
= & -q^{np}q^{-(s+n)t} : e(m+p-t)e(t) : +q^{-n(m+p)}q^{-(s-n)t} : e(m+p-t)e(t) : \\
& -q^{m(s-n)}q^{-(s-t)(m+t)} : e(p-t)e(m+t) : +q^{sm}q^{-(n+s)(m+t)} : e(p-t)e(m+t) : \\
& +\delta_{m+p,0}q^{np}q^{-(n+s)t}\big(\theta(-2t) - \theta(-2m-2t)\big) \\
& -\delta_{m+p,0}q^{-(s-n)t}\big(\theta(-2t) - \theta(-2m-2t)\big)
\end{aligned}
$$

by (2.36), we see that Proposition 2.10 and 2.12 hold true. ■

To find the correspondence of the homomorphism, we need to modify our operators by shifting some central elements.

For Proposition 2.3, we see that, if $n+s \in \Lambda(q)$ and $n-s \in \Lambda(q)$,

$$
\begin{aligned}
& [g_{ij}(m,n), h_{kl}(p,s)] \\
= & -\delta_{ik}q^{-n(m+p)}f_{jl}(m+p, s-n) - \delta_{jk}q^{np}f_{il}(m+p, n+s) \\
& -\delta_{il}q^{-(mn+np+ps)}f_{jk}(m+p, -(n+s)) - \delta_{jl}q^{(n-s)p}f_{ik}(m+p, n-s) \\
& +\delta_{ik}\delta_{jl}\delta_{m+p,0}m + \delta_{jk}\delta_{il}\delta_{m+p,0}q^{np}m.
\end{aligned}
$$

If $n+s \in \mathbb{Z} \setminus \Lambda(q)$ and $n-s \in \Lambda(q)$, then

$$
\begin{aligned}
& [g_{ij}(m,n), h_{kl}(p,s)] \\
= & -\delta_{ik}q^{-n(m+p)}f_{jl}(m+p, s-n) - \delta_{jl}q^{(n-s)p}f_{ik}(m+p, n-s) \\
& -\delta_{jk}q^{np}\Big(f_{il}(m+p, n+s) + \frac{1}{2}\delta_{il}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1}\Big) \\
& -\delta_{il}q^{-(mn+np+ps)}\Big(f_{jk}(m+p, -n-s) + \frac{1}{2}\delta_{jk}\delta_{m+p,0}\frac{q^{-n-s}+1}{q^{-n-s}-1})\Big) \\
& +\delta_{ik}\delta_{jl}\delta_{m+p,0}m.
\end{aligned}
$$

Similarly, if $n+s\in\Lambda(q)$ and $n-s\in\mathbb{Z}\setminus\Lambda(q)$

$$\begin{aligned}
& [g_{ij}(m,n),h_{kl}(p,s)] \\
= \; & -\delta_{jk}q^{np}f_{il}(m+p,n+s)-\delta_{il}q^{-(mn+np+ps)}f_{jk}(m+p,-n-s) \\
& -\delta_{ik}q^{-n(m+p)}\Big(f_{jl}(m+p,s-n)+\frac{1}{2}\delta_{jl}\delta_{m+p,0}\frac{q^{s-n}+1}{q^{s-n}-1}\Big) \\
& -\delta_{jl}q^{(n-s)p}\Big(f_{ik}(m+p,n-s)+\frac{1}{2}\delta_{ik}\delta_{m+p,0}\frac{q^{n-s}+1}{q^{n-s}-1})\Big) \\
& +\delta_{jk}\delta_{il}\delta_{m+p,0}q^{np}m.
\end{aligned}$$

By the above two relations, we have if $n+s,n-s\in\mathbb{Z}\setminus\Lambda(q)$

$$\begin{aligned}
& [g_{ij}(m,n),h_{kl}(p,s)] \\
= \; & -\delta_{jk}q^{np}\Big(f_{il}(m+p,n+s)+\frac{1}{2}\delta_{il}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1}\Big) \\
& -\delta_{il}q^{-(mn+np+ps)}\Big(f_{jk}(m+p,-n-s)+\frac{1}{2}\delta_{jk}\delta_{m+p,0}\frac{q^{-n-s}+1}{q^{-n-s}-1})\Big) \\
& -\delta_{ik}q^{-n(m+p)}\Big(f_{jl}(m+p,s-n)+\frac{1}{2}\delta_{jl}\delta_{m+p,0}\frac{q^{s-n}+1}{q^{s-n}-1}\Big) \\
& -\delta_{jl}q^{(n-s)p}\Big(f_{ik}(m+p,n-s)+\frac{1}{2}\delta_{ik}\delta_{m+p,0}\frac{q^{n-s}+1}{q^{n-s}-1})\Big)
\end{aligned}$$

Using the same method, for Proposition 2.5 we have, if $n+s\in\Lambda(q)$,

$$\begin{aligned}
& [f_{ij}(m,n),f_{kl}(p,s)] \\
= \; & \delta_{jk}q^{np}f_{il}(m+p,n+s)-\delta_{il}q^{sm}f_{kj}(m+p,n+s)-\delta_{jk}\delta_{il}q^{np}\delta_{m+p,0}m.
\end{aligned}$$

If $n+s\in\mathbb{Z}\setminus\Lambda(q)$, then

$$\begin{aligned}
& [f_{ij}(m,n),f_{kl}(p,s)] \\
= \; & \delta_{jk}q^{np}\Big(f_{il}(m+p,n+s)+\frac{1}{2}\delta_{il}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1}\Big) \\
& -\delta_{il}q^{sm}\Big(f_{kj}(m+p,n+s)+\frac{1}{2}\delta_{jk}\delta_{m+p,0}\frac{q^{n+s}+1}{q^{n+s}-1})\Big).
\end{aligned}$$

For Proposition 2.10, if $n-s\in\Lambda(q)$,

$$\begin{aligned}
& \{e_i(m,n),e_k^*(p,s)\}_+ \\
= \; & \tau\delta_{ik}q^{-n(m+p)}e_0(m+p,s-n)+\tau q^{p(n-s)}f_{ik}(m+p,n-s)-\tau\delta_{ik}\delta_{m+p,0}m.
\end{aligned}$$

If $n+s \in \mathbb{Z} \setminus \Lambda(q)$, then

$$\begin{aligned}
& \{e_i(m,n), e_k^*(p,s)\}_+ \\
= \;& \tau \delta_{ik} q^{-n(m+p)} \Big(e_0(m+p, s-n) + \frac{1}{2} \delta_{m+p,0} \frac{q^{s-n}+1}{q^{s-n}-1} \Big) \\
& + \tau q^{p(n-s)} \Big(f_{ik}(m+p, n-s) + \frac{1}{2} \delta_{jk} \delta_{m+p,0} \frac{q^{n-s}+1}{q^{n-s}-1}) \Big).
\end{aligned}$$

For Proposition 2.12, if $n+s \in \Lambda(q)$ and $n-s \in \Lambda(q)$,

$$\begin{aligned}
& [e_0(m,n), e_0(p,s)] \\
= \;& -(q^{np} - q^{sm}) e_0(m+p, n+s) - (q^{m(s-n)} - q^{-n(m+p)}) e_0(m+p, s-n) \\
& + \delta_{m+p,0} q^{np} m - \delta_{m+p,0} m.
\end{aligned}$$

If $n+s \in \mathbb{Z} \setminus \Lambda(q)$ and $n-s \in \Lambda(q)$, then

$$\begin{aligned}
& [e_0(m,n), e_0(p,s)] \\
= \;& -(q^{np} - q^{sm}) \Big(e_0(m+p, n+s) + \frac{1}{2} \delta_{m+p,0} \frac{q^{n+s}+1}{q^{n+s}-1} \Big) \\
& -(q^{m(s-n)} - q^{-n(m+p)}) e_0(m+p, s-n) - \delta_{m+p,0} m.
\end{aligned}$$

If $n+s \in \Lambda(q)$ and $n-s \in \mathbb{Z} \setminus \Lambda(q)$

$$\begin{aligned}
& [e_0(m,n), e_0(p,s)] \\
= \;& -(q^{np} - q^{sm}) e_0(m+p, n+s) + \delta_{m+p,0} q^{np} m \\
& -(q^{m(s-n)} - q^{-n(m+p)}) \Big(e_0(m+p, s-n) + \frac{1}{2} \delta_{m+p,0} \frac{q^{s-n}+1}{q^{s-n}-1} \Big).
\end{aligned}$$

If $n+s, n-s \in \mathbb{Z} \setminus \Lambda(q)$

$$\begin{aligned}
& [e_0(m,n), e_0(p,s)] \\
= \;& -(q^{np} - q^{sm}) \Big(e_0(m+p, n+s) + \frac{1}{2} \delta_{m+p,0} \frac{q^{n+s}+1}{q^{n+s}-1} \Big) \\
& -(q^{m(s-n)} - q^{-n(m+p)}) \Big(e_0(m+p, s-n) + \frac{1}{2} \delta_{m+p,0} \frac{q^{s-n}+1}{q^{s-n}-1} \Big).
\end{aligned}$$

Therefore, if we define

$$F_{ij}(m,n) = \begin{cases} f_{ij}(m,n), & \text{for } n \in \Lambda(q) \\ f_{ij}(m,n) + \frac{1}{2}\delta_{ij}\delta_{m,0}\frac{q^n+1}{q^n-1}, & \text{for } n \in \mathbb{Z} \setminus \Lambda(q) \end{cases}$$

$$G_{ij}(m,n) = g_{ij}(m,n), \quad H_{ij}(m,n) = h_{ij}(m,n), \tag{2.41}$$

$$E_i(m,n) = e_i(m,n), \quad E_i^*(m,n) = e_i^*(m,n),$$

$$E_0(m,n) = \begin{cases} e_0(m,n), & \text{for } n \in \Lambda(q) \\ e_0(m,n) + \frac{1}{2}\delta_{m,0}\frac{q^n+1}{q^n-1}, & \text{for } n \in \mathbb{Z} \setminus \Lambda(q) \end{cases}$$

we have

Theorem 2.1 *$V_\tau(N)$ is a module for the Lie superalgebra $\widehat{\mathcal{G}}$ under the action given by (for $\tau = \pm 1$)*

$$\begin{aligned} \pi(\tilde{g}_{ij}(m,n)) &= \tau G_{ij}(m,n), & \pi(\tilde{f}_{ij}(m,n)) &= F_{ij}(m,n), \\ \pi(\tilde{h}_{ij}(m,n)) &= \tau H_{ij}(m,n), & \pi(\tilde{e}_i(m,n)) &= \tau E_i(m,n), \\ \pi(\tilde{e}_i^*(m,n)) &= E_i^*(m,n), & \pi(\tilde{e}_0(m,n)) &= E_0(m,n), \\ \pi(c_x) &= -\frac{1}{2}, & \pi(c_y) &= 0. \end{aligned}$$

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